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Browder spectra of upper-triangular operator matrices ☆

Hai-Yan Zhang *, Hong-Ke Du

College of Mathematics and Information Science, Shaanxi Normal University, Xi'an 710062, PR China

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Abstract

Let $M_C = \left[\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right]$ be a 2×2 upper triangular operator matrix acting on the Hilbert space $\mathcal{H} \oplus \mathcal{K}$. In this paper, for given operators A and B, we prove that

$$\bigcap_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_b(M_C) = \left(\bigcap_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma(M_C)\right) \Big\backslash \big(\rho_b(A) \cap \rho_b(B)\big),$$

where $\rho_b(T) = \mathbb{C} \setminus \sigma_b(T)$ denotes the Browder resolvent of an operator T and $\bigcap_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma(M_C)$ has been determined in [H.K. Du, P. Jin, Perturbation of spectrums of 2×2 operator matrices, Proc. Amer. Math. Soc. 121 (1994) 761–776]. Moreover, we explore the relations of $\sigma(A) \cup \sigma(B) \setminus \sigma(M_C)$, $\sigma_b(A) \cup \sigma_b(B) \setminus \sigma_b(M_C)$ and $\sigma_w(A) \cup \sigma_w(B) \setminus \sigma_w(M_C)$, where $\sigma(A)$, $\sigma_b(A)$ and $\sigma_w(A)$ denote the spectrum, the Browder spectrum and the Weyl spectrum of A, respectively.

Keywords: Browder spectrum; Weyl spectrum; Upper-triangular operator matrix

1. Introduction

Throughout this paper, let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, $\mathcal{B}(\mathcal{K},\mathcal{H})$ denote the set of all bounded linear operators from $\mathcal{K}$ into $\mathcal{H}$ and abbreviate $\mathcal{B}(\mathcal{H},\mathcal{H})$ to $\mathcal{B}(\mathcal{H})$. If $A \in \mathcal{B}(\mathcal{H})$, write $N(A)$ and $R(A)$ for the null space and the range of A, respectively. If A is a semi-Fredholm operator and $n(A) = \dim N(A)$ and $d(A) = \dim(\mathcal{H}/\overline{R(A)})$, we define the index of A by $\operatorname{ind}(A) =$

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* Corresponding author.
E-mail addresses: csqam@163.com (H.Y. Zhang), hkdu@snnu.edu.cn (H.K. Du).

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$n(A) - d(A)$. $\mathcal{F}_r(\mathcal{H})$ and $\mathcal{F}_l(\mathcal{H})$ denote the sets of all the right and left semi-Fredholm operators, respectively [4]. For $A \in \mathcal{B}(\mathcal{H})$, A is called a Weyl operator if it is a Fredholm operator and its Fredholm index is zero. A is called a Browder operator if it is a Fredholm operator with finite ascent and finite descent. We write $\alpha(A)$ and $\beta(A)$ for the ascent and the descent of A, respectively. A is called Drazin invertible if there exists an operator $A^D \in \mathcal{B}(\mathcal{H})$ such that

$$AA^D = A^D A, \qquad A^D A A^D = A^D, \qquad A^{k+1} A^D = A^k,$$

for some nonnegative integer k. The least k in the previous definition is known as the Drazin index $i(A)$ of A [5]. As well known, A is Drazin invertible if and only if A has a finite ascent and a finite descent, in this case, $\alpha(A) = \beta(A) = i(A) < \infty$ [10]. Thus A is a Browder operator if and only if it is a Drazin invertible Fredholm operator. Denote $\sigma(A)$, $\sigma_r(A)$, $\sigma_l(A)$, $\sigma_p(A)$ and $\sigma_{ap}(A)$ for the spectrum, the right spectrum, the left spectrum, the point spectrum and the approximation point spectrum of A, respectively. Write $\operatorname{iso} F$ and ∂F for the set of all the isolated points and the boundary of $F \subset \mathbb{C}$, respectively. The essential spectrum $\sigma_e(A)$, the Weyl spectrum $\sigma_w(A)$, the Drazin spectrum $\sigma_D(A)$, the Browder spectrum $\sigma_b(A)$ and the deficiency spectrum $\sigma_\delta(A)$ [6] of A are defined by: $\sigma_e(A) = \{\lambda \in \mathrm{C}\colon\ A - \lambda I \text{ is not Fredholm}\}$, $\sigma_w(A) = \{\lambda \in \mathrm{C}\colon\ A - \lambda I \text{ is not Weyl}\}$, $\sigma_D(A) = \{\lambda \in \mathrm{C}\colon\ A - \lambda I \text{ is not Drazin invertible}\}$ (see [12]), $\sigma_b(A) = \{\lambda \in \mathrm{C}\colon\ A - \lambda I \text{ is not Browder}\}$ and $\sigma_\delta(A) = \{\lambda \in \mathbb{C}\colon\ R(A - \lambda) \text{ is not surjective}\}$, respectively. The residual spectrum [11] of A is defined by $\sigma_\gamma(A) = \{\lambda \in \mathbb{C}\colon\ N(A - \lambda) = 0 \text{ and } \overline{R(A - \lambda)} \neq \mathcal{H}\}$. Write $\sigma_{le}(A)$ and $\sigma_{re}(A)$ for the left and the right essential spectrum of A, respectively. Let $\sigma'_\gamma(A) = \{\lambda \in \sigma(A) \text{ and } \lambda \notin \sigma_{ap}(A)\}$ and $\sigma'_p(B) = \{\lambda \in \sigma(B) \text{ and } \lambda \notin \sigma_\delta(B)\}$. It is clear that $\sigma'_\gamma(A) \subset \sigma_\gamma(A)$, $\sigma'_p(B) \subset \sigma_p(B)$. Define $\Lambda_{(A,B)}$ by

$$\Lambda_{(A,B)} := \sigma'_\gamma(A) \cap \sigma'_p(B) \cap \bigl\{\lambda \in \mathbb{C}\colon n(B - \lambda) = d(A - \lambda)\bigr\}. \tag{1}$$

When $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ are given, we denote by M_C an operator acting on $\mathcal{H} \oplus \mathcal{K}$ of the form,

$$M_C = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}, \tag{2}$$

where $C \in B(K, H)$. In the sequel, M_C has the form as (2).

In the last decades considerable attention has been paid to upper triangular operator matrices [1–3,5–9,12]. For given $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$, the sets $\bigcap_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_\tau(M_C)$ were studied in some works [5,6,9,12], where $\sigma_\tau(M_C)$ can be equal to the spectrum, the left (right) essential, essential, Weyl or Drazin spectrum of M_C. For example, in [6], H. Du and J. Pan have proved that,

$$\bigcap_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma(M_C) = \sigma_{ap}(A) \cup \sigma_\delta(B) \cup \bigl\{\lambda \in \mathbb{C}\colon n(B - \lambda) \neq d(A - \lambda)\bigr\}. \tag{3}$$

In [5], D.S. Djordjević has obtained that

$$\bigcap_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_e(M_C) = \sigma_{le}(A) \cup \sigma_{re}(B) \cup W(A, B), \tag{4}$$

where $W(A, B) = \{\lambda \in \mathrm{C}\colon \dim R(A - \lambda)^\perp \neq \dim N(B - \lambda) \text{ and one of them is infinite}\}$. In [12], we have gotten

$$\bigcap_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_D(M_C) = \Bigl(\bigcap_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma(M_C)\Bigr) \Bigm\backslash \bigl(\rho_D(A) \cap \rho_D(B)\bigr). \tag{5}$$

In this paper, we shall prove that

$$\begin{aligned}&\bigcap_{C\in\mathcal{B}(\mathcal{K},\mathcal{H})}\sigma_b(M_C)\\&\quad=\big(\sigma_{ap}(A)\cup\sigma_\delta(B)\cup\big\{\lambda\in\mathbb{C}\colon n(B-\lambda)\neq d(A-\lambda)\big\}\big)\setminus\big(\rho_b(A)\cap\rho_b(B)\big),\end{aligned}$$

that is,

$$\bigcap_{C\in\mathcal{B}(\mathcal{K},\mathcal{H})}\sigma_b(M_C)=\Big(\bigcap_{C\in\mathcal{B}(\mathcal{K},\mathcal{H})}\sigma(M_C)\Big)\Big\backslash\big(\rho_b(A)\cap\rho_b(B)\big).$$

Moreover, we obtain that

$$\big(\sigma(A)\cup\sigma(B)\big)\setminus\sigma(M_C)=\big(\sigma_b(A)\cup\sigma_b(B)\big)\setminus\sigma_b(M_C)\subset\big(\sigma_w(A)\cup\sigma_w(B)\big)\setminus\sigma_w(M_C).$$

2. Main results and proofs

To prove the main results of this paper, we begin with some notations and terminology.

In [8], W.Y. Lee has obtained the following result, for given $A\in\mathcal{B}(\mathcal{H})$ and $B\in\mathcal{B}(\mathcal{K})$,

$$\sigma_w(A)\cup\sigma_w(B)=\sigma_w(M_C)\cup W_4(A,B,C),\tag{6}$$

where $W_4(A,B,C)$ is the union of certain of the holes in $\sigma_w(M_C)$ which happen to be subsets of $\sigma_w(A)\cap\sigma_w(B)$. In [7], J.K. Han et al. have shown that

$$\sigma(A)\cup\sigma(B)=\sigma(M_C)\cup W_1(A,B,C),\tag{7}$$

where $W_1(A,B,C)$ is the union of certain of the holes in $\sigma(M_C)$ which happen to be subsets of $\sigma(A)\cap\sigma(B)$.

For A and B in $\mathcal{B}(\mathcal{H})$, if $\{n_i\}$ is an increase of nonnegative integers with $n_i\neq n_j$ if $i\neq j$, denote U_{n_k} by $U_{n_0}=\emptyset$ and

$$\begin{aligned}U_{n_k}=\big\{\lambda\in\sigma(A)\cap\sigma(B)\colon\ &n(B-\lambda)=d(A-\lambda)=n_k,\ n_k\in\mathbb{N}\cup\{+\infty\},\\&N(A-\lambda)=\{0\},\ R(A-\lambda)\text{ is closed and }R(B-\lambda)=\mathcal{K}\big\}\end{aligned}$$

for $k\geqslant 1$, respectively. It is clear that $U_{n_i}\neq U_{n_j}$ if $i\neq j$.

Next, we shall give some lemmas.

Lemma 1. *Given $A\in\mathcal{B}(\mathcal{H})$, $B\in\mathcal{B}(\mathcal{K})$, then U_{n_k} is open and*

$$\Lambda_{(A,B)}=\sigma(A)\cup\sigma(B)\Big\backslash\bigcap_{C\in\mathcal{B}(\mathcal{K},\mathcal{H})}\sigma(M_C)=\bigcup_{k=0}^{n}U_{n_k}\quad\big(n\in\mathbb{N}\cup\{+\infty\}\big),$$

where $\Lambda_{(A,B)}$, M_C, U_{n_k} are defined above.

Proof. For any two operators C_1 and C_2 in $\mathcal{B}(\mathcal{K},\mathcal{H})$, from (7), we have

$$\sigma(M_{C_1})\cup W_1(A,B,C_1)=\sigma(A)\cup\sigma(B)$$

and

$$\sigma(M_{C_2})\cup W_1(A,B,C_2)=\sigma(A)\cup\sigma(B),$$

where $\sigma(M_{C_i}) \cap W_1(A,B,C_i) = \emptyset$ $(i = 1,2)$. It is clear that

$$\bigl\{\bigl(W_1(A,B,C_1) \cap W_1(A,B,C_2)\bigr) \cup \bigl(\sigma(M_{C_1}) \cap W_1(A,B,C_2)\bigr) \cup \bigl(\sigma(M_{C_2}) \cap W_1(A,B,C_1)\bigr)\bigr\} \subset W_1(A,B,C_1) \cup W_1(A,B,C_2).$$

On the other hand, for any $\lambda \in W_1(A,B,C_1) \cup W_1(A,B,C_2)$, without loss of generality, we suppose $\lambda \in W_1(A,B,C_1)$. If $\lambda \in W_1(A,B,C_2)$, then $\lambda \in W_1(A,B,C_1) \cap W_1(A,B,C_2)$. If $\lambda \notin W_1(A,B,C_2)$, then $\lambda \in \sigma(M_{C_2})$, so $\lambda \in \sigma(M_{C_2}) \cap W_1(A,B,C_1)$. Hence, $\lambda \in (W_1(A,B,C_1) \cap W_1(A,B,C_2)) \cup (\sigma(M_{C_1}) \cap W_1(A,B,C_2)) \cup (\sigma(M_{C_2}) \cap W_1(A,B,C_1))$, so

$$\begin{aligned} &W_1(A,B,C_1) \cup W_1(A,B,C_2) \\ &\quad \subset \bigl\{\bigl(W_1(A,B,C_1) \cap W_1(A,B,C_2)\bigr) \cup \bigl(\sigma(M_{C_1}) \cap W_1(A,B,C_2)\bigr) \\ &\qquad \cup \bigl(\sigma(M_{C_2}) \cap W_1(A,B,C_1)\bigr)\bigr\}. \end{aligned}$$

Thus

$$\begin{aligned} &W_1(A,B,C_1) \cup W_1(A,B,C_2) \\ &\quad = \bigl\{\bigl(W_1(A,B,C_1) \cap W_1(A,B,C_2)\bigr) \cup \bigl(\sigma(M_{C_1}) \cap W_1(A,B,C_2)\bigr) \\ &\qquad \cup \bigl(\sigma(M_{C_2}) \cap W_1(A,B,C_1)\bigr)\bigr\}. \end{aligned}$$

Therefore

$$\begin{aligned} &\bigl(\sigma(M_{C_1}) \cap W_1(A,B,C_1)\bigr) \cup \bigl(\sigma(M_{C_2}) \cap W_1(A,B,C_2)\bigr) \\ &\quad = \bigl(\sigma(M_{C_1}) \cap \sigma(M_{C_2})\bigr) \cup \bigl(W_1(A,B,C_1) \cup W_1(A,B,C_2)\bigr) = \sigma(A) \cup \sigma(B) \end{aligned}$$

and $(\sigma(M_{C_1}) \cap \sigma(M_{C_2})) \cap (W_1(A,B,C_1) \cup W_1(A,B,C_2)) = \emptyset$. Since the choices of C_1 and C_2 are arbitrary,

$$\bigcap_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma(M_C) \cup \Biggl(\bigcup_{C \in B(\mathcal{K},\mathcal{H})} W_1(A,B,C)\Biggr) = \sigma(A) \cup \sigma(B), \tag{8}$$

and

$$\bigcap_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma(M_C) \cap \Biggl(\bigcup_{C \in B(\mathcal{K},\mathcal{H})} W_1(A,B,C)\Biggr) = \emptyset. \tag{9}$$

So

$$\sigma(A) \cup \sigma(B) \Bigm\backslash \bigcap_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma(M_C) = \bigcup_{C \in B(\mathcal{K},\mathcal{H})} W_1(A,B,C).$$

From the definition of $\Lambda_{(A,B)}$, formulae (1) and (3), we get

$$\Lambda_{(A,B)} = \sigma(A) \cup \sigma(B) \Bigm\backslash \bigcap_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma(M_C) = \bigcup_{C \in B(\mathcal{K},\mathcal{H})} W_1(A,B,C).$$

For any $\lambda \in \sigma'_\gamma(A)$, by the definition of $\sigma'_\gamma(A)$, then $R(A-\lambda)$ is closed and $N(A-\lambda) = \{0\}$. Thus $\operatorname{ind}(A-\lambda) = -n$ or $-\infty$, hence $\{A-\lambda\colon \lambda \in \sigma'_\gamma(A)\}$ is a subset of an union of some components of $\mathcal{F}_l(\mathcal{H})$. Similarly, $\{B-\lambda\colon \lambda \in \sigma'_p(B)\}$ is a subset of an union of some components of $\mathcal{F}_r(\mathcal{K})$. Because that each component of $\mathcal{F}_r(\mathcal{K})$ (or $\mathcal{F}_l(\mathcal{H})$) is connected and $W_1(A,B,C)$ is the union of certain of holes for any $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$, therefore $\Lambda_{(A,B)} = \bigcup_{k=0}^{n} U_{n_k}$ $(n \in \mathbb{N} \cup \{+\infty\})$ and each U_{n_k} is open. So the result can be obtained. $\square$

Let $A \in \mathcal{B}(\mathcal{K},\mathcal{H})$. By the definitions of the Browder operator and the Drazin operator, we have $\sigma_b(A) = \sigma_D(A) \cup \sigma_e(A)$. Using the relation, we investigate $\bigcap_{C\in\mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_b(M_C)$.

Lemma 2. *For given operators $A \in \mathcal{B}(\mathcal{H})$, $B \in \mathcal{B}(\mathcal{K})$, then*

$$\sigma_b(A) \cup \sigma_b(B) = \sigma_b(M_C) \cup W_3(A, B, C), \tag{10}$$

where $W_3(A, B, C)$ is the union of certain of the holes in $\sigma_b(M_C)$ which happen to be subsets of $\sigma_b(A) \cap \sigma_b(B)$.

Proof. By (4), we have

$$\partial\big(\sigma_e(A) \cup \sigma_e(B)\big) \subset \partial\sigma_e(A) \cup \partial\sigma_e(B) \subset \sigma_{le}(A) \cup \sigma_{re}(B) \subset \sigma_e(M_c).$$

Thus

$$\begin{aligned}
\partial\big(\sigma_b(A) \cup \sigma_b(B)\big) &= \partial\big(\sigma_e(A) \cup \sigma_D(A) \cup \sigma_e(B) \cup \sigma_D(B)\big) \\
&\subset \partial\big(\sigma_e(A) \cup \sigma_e(B)\big) \cup \partial\big(\sigma_D(A) \cup \sigma_D(B)\big) \\
&\subset \sigma_e(M_C) \cup \sigma_D(M_C) \\
&= \sigma_b(M_C).
\end{aligned}$$

On the other hand, $\sigma_b(M_C) \subset \sigma_b(A) \cup \sigma_b(B)$. Hence

$$\eta\big(\sigma_b(M_C)\big) = \eta\big(\sigma_b(A) \cup \sigma_b(B)\big) \quad [2], \tag{11}$$

where $\eta(\cdot)$ denotes the polynomially convex hull. Equality (11) says that the passage from $\sigma_b(M_C)$ to $\sigma_b(A) \cup \sigma_b(B)$ in certain of the holes in $\sigma_b(M_C)$. Since $(\sigma_b(A) \cup \sigma_b(B)) \setminus \sigma_b(M_C) \subset \sigma_b(A) \cap \sigma_b(B)$, the filling some holes in $\sigma_b(M_C)$ should occur in $\sigma_b(A) \cap \sigma_b(B)$. □

Lemma 3. *For $A \in \mathcal{B}(\mathcal{H})$, then $\sigma(A) \setminus \sigma_b(A) \subset \operatorname{iso}\sigma(A)$.*

Proof. Combining $\sigma(A) \setminus \sigma_D(A) \subset \operatorname{iso}\sigma(A)$ with $\sigma_b(A) = \sigma_e(A) \cup \sigma_D(A)$, the result holds. □

Theorem 4. *For given $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$, we have*

$$\begin{aligned}
&\bigcap_{C\in\mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_b(M_C) \\
&\quad = \big(\sigma_{ap}(A) \cup \sigma_\delta(B) \cup \big\{\lambda \in \mathbb{C}\colon\ n(B-\lambda) \neq d(A-\lambda)\big\}\big) \setminus \big(\rho_b(A) \cup \rho_b(B)\big).
\end{aligned}$$

Proof. For convenience, we divide the proof into three steps.

Step 1. We shall prove that if $\lambda \in (\operatorname{iso}\sigma(A) \cap \sigma_b(A)) \cup (\operatorname{iso}\sigma(B) \cap \sigma_b(B))$, then $\lambda \in \sigma_b(M_C)$ for all $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$. Without loss of generality, we suppose $\lambda \in \operatorname{iso}\sigma(A) \cap \sigma_b(A)$, according to $\sigma_b(A) = \sigma_e(A) \cup \sigma_D(A)$, thus $\operatorname{iso}\sigma(A) \cap \sigma_b(A) = (\operatorname{iso}\sigma(A) \cap \sigma_e(A)) \cup (\operatorname{iso}\sigma(A) \cap \sigma_D(A))$. So $\lambda \in \operatorname{iso}\sigma(A) \cap \sigma_e(A)$ or $\lambda \in \operatorname{iso}\sigma(A) \cap \sigma_D(A)$. If $\lambda \in \operatorname{iso}\sigma(A) \cap \sigma_e(A)$, from (4), $\lambda \in \sigma_e(M_C)$ for all $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$. If $\lambda \in \operatorname{iso}\sigma(A) \cap \sigma_D(A)$, then $\lambda \in \sigma_D(M_C)$ [12, Theorem 2] for all $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$. Hence $\lambda \in \sigma_b(M_C)$.

Step 2. Let $E_1 = \sigma(A) \setminus \sigma_b(A)$, $E_2 = \sigma(B) \setminus \sigma_b(B)$, $E_0 = \sigma(M_C) \setminus \sigma_b(M_C)$. By Lemma 3, $E_1 \subset \operatorname{iso}\sigma(A)$, $E_2 \subset \operatorname{iso}\sigma(B)$ and $E_0 \subset \operatorname{iso}\sigma(M_C)$. And (7) implies that

$$\operatorname{iso}\sigma(M_C) = \operatorname{iso}\big(\sigma(A) \cup \sigma(B)\big) \subset \operatorname{iso}\sigma(A) \cup \operatorname{iso}\sigma(B). \tag{12}$$

It is clear that $E_1 \cap E_2 \subset E_0$, $E_1 \cap \rho(B) \subset E_0$ and $E_2 \cap \rho(A) \subset E_0$. Hence

$$(E_1 \cap E_2) \cup \big(E_1 \cap \rho(B)\big) \cup \big(E_2 \cap \rho(A)\big) \subset E_0.$$

On the other hand, for any $\lambda \in E_0$, then $\lambda \in \operatorname{iso}\sigma(M_C)$. Thus $\lambda \in \operatorname{iso}\sigma(A) \cup \operatorname{iso}\sigma(B)$ by (12). Without loss of generality, suppose $\lambda \in \operatorname{iso}\sigma(A)$, from Step 1, $\lambda \in E_1$. If $\lambda \in E_2$, then $\lambda \in E_2 \cap E_1$. If $\lambda \notin E_2$, then $\lambda \in \rho(B)$. So

$$E_0 \subset (E_1 \cap E_2) \cup \big(E_1 \cap \rho(B)\big) \cup \big(E_2 \cap \rho(A)\big).$$

Therefore,

$$E_0 = (E_1 \cap E_2) \cup \big(E_1 \cap \rho(B)\big) \cup \big(E_2 \cap \rho(A)\big). \tag{13}$$

By the definitions of E_1 and E_2, we have $E_0 \cap (\sigma_b(A) \cup \sigma_b(B)) = \emptyset$ and $(E_1 \cup E_2) \setminus E_0 \subset \sigma_b(A) \cup \sigma_b(B)$. Thus

$$\sigma(A) \cup \sigma(B) = \sigma_b(A) \cup \sigma_b(B) \cup E_0 \cup \big((E_1 \cup E_2) \setminus E_0\big) = \sigma_b(A) \cup \sigma_b(B) \cup E_0.$$

Thus

$$\big(\sigma(A) \cup \sigma(B)\big) \setminus \big(\sigma_b(A) \cup \sigma_b(B)\big) = E_0. \tag{14}$$

Step 3. In this step we first prove that $W_3(A,B,C) = W_1(A,B,C)$, where $W_1(A,B,C)$ and $W_3(A,B,C)$ are denoted as in (7) and (10), respectively.

$$\begin{aligned}\sigma(A) \cup \sigma(B) &= \sigma(M_C) \cup W_1(A,B,C)\\ &= \sigma_b(M_C) \cup E_0 \cup W_1(A,B,C)\\ &= \sigma_b(A) \cup \sigma_b(B) \cup E_1 \cup E_2\\ &= \sigma_b(M_C) \cup E_1 \cup E_2 \cup W_3(A,B,C).\end{aligned}$$

And according to $((E_1 \cup E_2) \setminus E_0) \subset \sigma_b(M_C)$ and (10), we have $W_3(A,B,C) \cap (\sigma_b(M_C) \cup (E_1 \cup E_2)) = \emptyset$ and $\sigma(A) \cup \sigma(B) = \sigma_b(M_C) \cup E_0 \cup W_3(A,B,C)$. Moreover, $W_1(A,B,C) \cap \sigma(M_C) = \emptyset$, so

$$W_3(A,B,C) = W_1(A,B,C). \tag{15}$$

Since equality (15) holds for any C,

$$\bigcup_{C \in B(\mathcal{K},\mathcal{H})} W_3(A,B,C) = \bigcup_{C \in B(\mathcal{K},\mathcal{H})} W_1(A,B,C). \tag{16}$$

Similar to formulae (8) and (9), we have

$$\bigcap_{C \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_b(M_C) \cup \Bigg(\bigcup_{C \in B(\mathcal{K},\mathcal{H})} W_3(A,B,C)\Bigg) = \sigma_b(A) \cup \sigma_b(B) \tag{17}$$

and

$$\bigcap_{C\in\mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_b(M_C) \cap \Big(\bigcup_{C\in B(\mathcal{K},\mathcal{H})} W_3(A,B,C) \Big) = \emptyset, \tag{18}$$

respectively. Combining formulae (8), (9), (14) with formulae (16), (17) and (18), we get

$$\begin{aligned}\Big(\bigcap_{C\in\mathcal{B}(\mathcal{K},\mathcal{H})} \sigma(M_C) \Big) \Big\backslash \Big(\bigcap_{C\in\mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_b(M_C) \Big) &= \big(\sigma(A)\cup\sigma(B)\big) \setminus \big(\sigma_b(A)\cup\sigma_b(B)\big) \\ &= E_0.\end{aligned}$$

By the definitions of E_1 and E_2, we have $\rho_b(A) = E_1 \cup \rho(A)$ and $\rho_b(B) = E_2 \cup \rho(B)$. Thus from (13), we obtain that

$$\rho_b(A)\cap\rho_b(B) = (E_1\cap E_2)\cup\big(E_1\cap\rho(B)\big)\cup\big(E_2\cap\rho(A)\big)\cup\big(\rho(A)\cap\rho(B)\big) \tag{19}$$

$$= E_0 \cup \big(\rho(A)\cap\rho(B)\big). \tag{20}$$

On the other hand, it is clear that

$$E_0 \cap \big(\rho(A)\cap\rho(B)\big) = \emptyset \tag{21}$$

and

$$\big(\rho(A)\cap\rho(B)\big) \cap \Big(\bigcap_{C\in\mathcal{B}(\mathcal{K},\mathcal{H})} \sigma(M_C) \Big) = \emptyset. \tag{22}$$

Combining (3) with (19), (20) and (21), then

$$\begin{aligned}&\Big(\bigcap_{C\in\mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_b(M_C) \Big) \\ &\quad = \Big(\bigcap_{C\in\mathcal{B}(\mathcal{K},\mathcal{H})} \sigma(M_C) \Big) \Big\backslash E_0 \\ &\quad = \Big(\bigcap_{C\in\mathcal{B}(\mathcal{K},\mathcal{H})} \sigma(M_C) \Big) \Big\backslash \big(\rho_b(A)\cap\rho_b(B)\big) \\ &\quad = \big(\sigma_{ap}(A)\cup\sigma_\delta(B)\cup\big\{\lambda\in\mathbb{C}\colon n(B-\lambda)\neq d(A-\lambda)\big\}\big) \setminus \big(\rho_b(A)\cap\rho_b(B)\big).\end{aligned}$$

The proof is completed. □

It is well known that $\sigma_e(A)\subset\sigma_w(A)\subset\sigma_b(A)\subset\sigma(A)$ and $\sigma_D(A)\subset\sigma(A)$, where $A\in\mathcal{B}(\mathcal{H})$. But in general, there does not exist an inclusion between $\sigma_D(A)$ and $\sigma_w(A)$. For example, let

$$A = \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix},$$

where S is unilateral shift on $\mathcal{H}$. Then $0\in\sigma_w(A)$ but $0\notin\sigma_D(A)$ since $A^2=0$. In [12], we have obtained that $(\sigma(A)\cup\sigma(B))\setminus\sigma(M_C) = (\sigma_D(A)\cup\sigma_D(B))\setminus\sigma_D(M_C)$. In the proof of Theorem 4, we get that $(\sigma(A)\cup\sigma(B))\setminus\sigma(M_C) = (\sigma_b(A)\cup\sigma_b(B))\setminus\sigma_b(M_C)$. In the following theorem, we shall study the relation between $\sigma(A)\cup\sigma(B)\setminus\sigma(M_C)$ and $\sigma_w(A)\cup\sigma_w(B)\setminus\sigma_w(M_C)$.

Theorem 5. *$W_1(A,B,C)\subset W_4(A,B,C)$, where $W_1(A,B,C)$ and $W_4(A,B,C)$ are denoted as in* (7) *and* (6)*, respectively.*

Proof. For any $\lambda \in W_1(A, B, C)$, then $\lambda \in \Lambda_{(A,B)} = \bigcup_{k=0}^{n} U_{n_k}$ by Lemma 1. Thus there exists a nonnegative integer k, such that $\lambda \in U_{n_k}$, that is, $n(B-\lambda) = d(A-\lambda) = n_k$, $n_k \in \mathbb{N} \cup \{+\infty\}$, $N(A-\lambda) = 0$, $R(A-\lambda)$ is closed, and $R(B-\lambda) = \mathcal{K}$. Hence $\lambda \in \sigma_w(A) \cup \sigma_w(B)$ and $M_C - \lambda$ is invertible. So $M_C - \lambda$ is Weyl, thus $\lambda \in W_3(A, B, C)$.

On the other hand, it is possible that there exists a $\lambda_0 \in \sigma_w(A) \cup \sigma_w(B)$, such that $n(B - \lambda_0) = n < \infty$, $d(A-\lambda_0) = m < \infty$ $(m \neq n \neq 0)$, $n(A-\lambda_0) = m - n$ and $R(A-\lambda_0)$ is closed, $R(B-\lambda_0) = \mathcal{K}$. Then $B - \lambda_0$ and $A - \lambda_0$ are Fredholm operators with $\operatorname{ind}(B-\lambda_0) = n$ and $\operatorname{ind}(A-\lambda_0) = -n$, respectively. Thus $M_C - \lambda_0$ is a Fredholm operator with $\operatorname{ind}(M_C - \lambda_0) = 0$. So $M_C - \lambda_0$ is Weyl, that is, $\lambda_0 \in W_3(A, B, C)$. But $M_C - \lambda_0$ is not invertible, so $\lambda_0 \notin W_1(A, B, C)$. The proof is completed. □

From Theorem 5 and the proof of Theorem 4, we have a consequence as follows.

Corollary 6. *For given operators $A \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, if $\sigma_w(A) \cup \sigma_w(B) = \sigma_w(M_C)$ then the following statements hold*:

(1) $\sigma(A) \cup \sigma(B) = \sigma(M_C)$;
(2) $\sigma_D(A) \cup \sigma_D(B) = \sigma_D(M_C)$;
(3) $\sigma_b(A) \cup \sigma_b(B) = \sigma_b(M_C)$,

and (1)–(3) *are equivalent.*

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